First exercise sheet "Class field theory" summer term 2025.
Throughout the following let G be any group.

Problem 1 (5 points). *Let $\left(M_\lambda\right)_{\lambda\in\Lambda}$ be a family of G-modules. Let $\prod_{\lambda\in\Lambda} M_\lambda$ be the set of all families $\left(m_\lambda\right)_{\lambda\in\Lambda}$ such that $m_\lambda \in M_\lambda$ for all $\lambda \in \Lambda$, with group operation*

$$(m+\mu)_\lambda = m_\lambda + \mu_\lambda,$$

G-action

$$(gm)_\lambda = g(m_\lambda)$$

and projection $\pi_\lambda(m) = m_\lambda$ to M_λ. Show that the universal property of a product in the category of G-modules holds!

Problem 2 (6 points). *Let $\left(M_\lambda\right)_{\lambda\in\Lambda}$ be a family of G-modules. Let $C = \coprod_{\lambda\in\Lambda} M_\lambda$ be the set of all families $(m_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} M_\lambda$ for which $\{\lambda \in \Lambda | m_\lambda \neq 0\}$ is finite. Let $M_\lambda \xrightarrow{i_\lambda} C$ send $m \in M_\lambda$ to the family $(\mu_\vartheta)_{\vartheta\in\Lambda}$ where $\mu_\lambda = m$ and $\mu_\vartheta = 0$ when $\vartheta \neq \lambda$. Show that the universal property of a coproduct in the category of G-modules holds.*

Problem 3 (5 points). *Let L/K be a Galois extension of degree p of fields of characteristic $p > 0$. Let σ be a generator of* $\mathrm{Gal}(L/K)$.

- *Show that*

  $$0 \to K \to L \xrightarrow{x \to \sigma(x) - x} L \xrightarrow{\mathrm{Tr}_{L/K}} K \to 0$$

  is exact!
- *Show that there is $x \in L$ with $\sigma(x) = x + 1$!*
- *Show that $\xi = x^p - x \in K$ and that L is a field of decomposition of the polynomial $T^p - T - \xi$!*

Problem 4 (2 point). *Let K be a field of characteristic $p > 0$ and $\xi \in K$ such that $x^p - x = \xi$ has no solution $x \in K$. Let L be the extension of K obtained by adjoining a solution x to this equation. Show that L/K is a Galois extension!*

An extension of this type is called an *Artin-Schreier extension.*

Problem 5 (3 points). *Let L/K be an algebraic field extension and $\overline{K}$ an algebraic closure of K. Show that the following conditions are equivalent:*

- *If an irreducible $P \in K[T]$ has a zero in L then it splits into linear factors in $L[T]$.*
- *L is a union of splitting fields of polynomials in $K[T]$.*
- *Two arbitrary extensions $L \xrightarrow{\lambda, \vartheta} \overline{K}$ of the embedding $K \to \overline{K}$ have the same image.*

One of the 21 points from this sheet is a bonus point which does not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday April 15 24:00.